COLORING INFINITE GRAPHS AND THE BOOLEAN PRIME IDEAL THEOREM

BY

H. LÄUCHLI

ABSTRACT

It is shown that the following theorem holds in set theory without AC: There is a function G which assigns to each Boolean algebra B a graph $G(B)$ such that (1) if $G(B)$ is 3-colorable then there is a prime ideal in B and (2) every finite subgraph of $G(B)$ is 3-colorable. The proof uses a combinatorial lemma on finite graphs.

Consider the following statements:

I In every Boolean algebra there is a prime ideal

P_n If G is a graph such that every finite subgraph G^* of G is n-colorable then G itself is n-colorable.

C_n The Cartesian product of a family of sets which have n members each is non-empty.

The following implications are provable in set theory without the axiom of choice (AC):

$$I \rightarrow P_{n+1} \rightarrow P_n \rightarrow C_n$$

$$C_2 \rightarrow P_2$$

(see Mycielski [6] and [7]. For implications among the C_n's see Mostowski [5] and Gauntt [1]).

It is known that C_n is considerably weaker than I (see Lévy [4], the diagram on p. 224). In [3] Lévy shows that $C_n \rightarrow P_3$ is not provable for any n. Here we strengthen this result by proving $P_3 \rightarrow I$. Thus for $n \geqq 3$, the equivalence $P_n \leftrightarrow I$ is provable in set theory without AC.

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Many other equivalents of I are known; see [3] and [8] where other references are given. It should also be mentioned that I is implied by and properly weaker than AC (see Halpern [2]).

The proof of $P_3 \to I$ is given in two parts: In part 1 we prove an elementary combinatorial lemma which asserts the existence of certain finite graphs. In part 2 we show how each Boolean algebra B is associated with a graph $G(B)$ in such a way that the 3-colorings of $G(B)$ yield prime ideals of B, and the 3-colorings of the finite subgraphs of $G(B)$ are yielded by the prime ideals of the finite subalgebras of B.

Part 1

NOTATION. A graph is an ordered pair $G = \langle A, R \rangle$, where R is a symmetric, irreflexive binary relation on A. A is the set of vertices of G, denoted by $|G|$. R, the set of edges of G, is denoted by G°. G' is a subgraph of G, $G' \leqq G$, if $|G'| \subseteq |G|$ and $G'^\circ \subseteq G^\circ$. (Notice that two vertices of G' which are connected by an edge of G are not necessarily connected by an edge of G'). If N is a set of graphs then the sum of N, ΣN, is the graph $\langle A, R \rangle$ with $A = \bigcup\{|G|: G \in N\}$, $R = \bigcup\{G^\circ: G \in N\}$. We write $G + G'$ for $\Sigma\{G, G'\}$. $\bar{\bar{G}}$ denotes the cardinality of $|G|$. G is a complete n-graph, if $\bar{\bar{G}} = n$ and each pair of distinct vertices is joined by an edge. An n-coloring of a graph $\langle A, R \rangle$ is a function σ from A into n ($n = \{0, 1, \cdots, n-1\}$) such that $xRy \to \sigma x \neq \sigma y$, all $x, y \in A$. $C_n(G)$ denotes the set of all n-colorings of G. If r is an equivalence relation on a set E, $[r]$ denotes the corresponding partition of E. $eq_n(E)$ denotes the set of all equivalence relations r on E with $\overline{\overline{[r]}} \leqq n$. Thus $eq_n(E) \subseteq eq_{n+1}(E)$. We use $.. \restriction E$ ambiguously to denote the functional restriction $\sigma \restriction E$ of a function σ to a subset E of dom(σ), or to denote the relational restriction $r \restriction E$ of a relation r to E. $\sigma \parallel E$ denotes the equivalence relation on E induced by σ, i.e. $\sigma \parallel E = \{\langle x, y \rangle \in E \times E: \sigma x = \sigma y\}$. If $E \subseteq |G|$, let $R_n(G, E) = \{\sigma \parallel E: \sigma \in C_n(G)\}$. Thus $R_n(G, E)$ is the set of equivalence relations on E which can be extended to an n-coloring of G.

COLORING EXTENSION LEMMA. *Given $n \geqq 3$, a finite set E and any subset K of $eq_n(E)$. Then there exists a finite graph G, $E \subseteq |G|$, such that $R_n(G, E) = K$.*

COROLLARY. *Given a finite set E and $K \subseteq eq_2(E)$, there is a finite G, $E \subseteq |G|$, such that $R_3(G, E) = K$.*

The Corollary is all we need in proving $P_3 \to I$. We shall prove the general lemma since it might be of interest in its own.

PROPOSITION 1. *If* $E = |G_1| \cap |G_2|$, *then*

$$R_n(G_1 + G_2, E) = R_n(G_1, E) \cap R_n(G_2, E).$$

The proof is immediate. (Note that if $\tau \parallel E \in R_n(G, E)$ then there is $\sigma \in C_n(G)$ with $\sigma \upharpoonright E = \tau \upharpoonright E$).

In virtue of Proposition 1 it is sufficient to prove the Coloring Extension Lemma for each of the sets $K_r = eq_n(E) - \{r\}$, $r \in eq_n(E)$. For, let G_r be graphs appropriate for K_r and such that $|G_r| \cap |G_s| = E$ for $r \neq s$. Then the graph $\Sigma\ \{G_r\colon r \in eq_n(E) - K\}$ is appropriate for K if $K \neq eq_n(E)$. If $K = eq_n(E)$ we take the graph over E without any edges.*

PROPOSITION 2. *Let* $n \geqq 3$. *Then there are arbitrarily large finite graphs* G *such that*

i) G *is not* $(n-1)$*-colorable,*

ii) *given any non-constant function* f *from* $|G|$ *into* n, *there is an* n*-coloring* σ *of* G *such that* $\sigma x \neq fx$, *all* $x \in |G|$; *indeed for each* $h \geqq n$ *with* $h \equiv n \pmod 2$ *there is a graph* G, $\bar{\bar{G}} = h$, *satisfying* (i) *and* (ii).

Proof by induction on n. If $n = 3$, $h \geqq 3$, h odd, let G be a cycle with h vertices; say $G = \langle h, R \rangle$, where $0R1$, $1R2, \cdots (h-1)R0$. G is not 2-colorable since h is odd and $\geqq 3$. Given a non-constant function f from h into 3, there are adjacent vertices x, y with $fx \neq fy$, say $f0 \neq f(h-1)$. Let $\sigma 0 = f(h-1)$ and $\sigma(i+1) < 3$ such that $\sigma(i+1) \neq \sigma i$ and $\sigma(i+1) \neq f(i+1)$, $i = 0, 1, \cdots, h-2$. Then $\sigma 0 \neq f0$ since $f(h-1) \neq f0$, and $\sigma(h-1) \neq \sigma 0$ since $\sigma 0 = f(h-1)$. Therefore σ is a 3-coloring of G with the required property.

Induction step. Let $h \geqq n+1$, $h \equiv n+1 \pmod 2$. Let G be a graph, $\bar{\bar{G}} = h-1$, satisfying (i) and (ii) with respect to n. We introduce one new vertex Q and join it with each vertex of G be an edge. The resulting graph G' has h vertices and satisfies (i) and (ii) with respect to $n+1$. (i) is trivial.

PROOF OF (ii). Let f be a non-constant function from $|G'|$ into $n+1$. There is a permutation π on $n+1$ such that the function $f_1 = \pi \circ f$ satisfies (a) $f_1(Q) \neq n$ and (b) $f_1(x) = n$ for some $x \in |G|$. Since $h - 1 > 1$ and $n > 1$, (b) implies the existence of a non-constant function f_2 from $|G|$ into n such that $f_2 y = f_1 y$ whenever $f_1 y < n$. By induction hypothesis there is an n-coloring τ of G with

* Victor Harnik found a proof of the Coloring Extension Lemma which is dual to the proof given here: He starts with the trivial cases $K = \{r\}$ and gives for $n \geqq 3$ a non-trivial construction of a "graph multiplication" M_n such that

$$R_n(M_n(G_1, G_2), E) = R_n(G_1, E) \cup R_n(G_2, E).$$

$\tau y \neq f_2 y$, all $y \in |G|$. Since the range of τ is included in n, we have $\tau y \neq f_1 y$, all $y \in |G|$, and τ can be extended to an $(n+1)$-coloring of G' by setting $\tau Q = n$. By (a), $\tau y \neq f_1 y$ for all $y \in |G'|$. Thus $\sigma = \pi^{-1} \circ \tau$ is an $(n+1)$-coloring of G' such that $\sigma y \neq f y$, all $y \in |G'|$.

Note that Proposition 2 fails for $n = 2$. Except in reference to this proposition, the hypothesis $n > 2$ will not be used any more in the first part.

For the following graph constructions we fix $n \geqq 3$ and consider triples $\langle G, E, r \rangle$ where G is a finite graph, $E \subseteq |G|$ and $r \in eq_n(E)$. Let "ext" be the relation defined by $\langle G, E, r \rangle \operatorname{ext} \langle G', E', r' \rangle$ iff $G \leqq G'$ and for all σ:

(a) if $\sigma \in C_n(G')$ and $\sigma \parallel E = r$ then $\sigma \parallel E' = r'$,

(b) if $\sigma \in C_n(G)$ and $\sigma \parallel E \neq r$ then there is $\tau \in C_n(G')$ such that $\sigma \subseteq \tau$ and $\tau \parallel E' \neq r'$.

The relation "ext" is transitive and reflexive. We consider the following conditions on a triple $\langle G, E, r \rangle$:

(C_0) no condition

(C_1) $\overline{\overline{[r]}} = n$

(C_2) $\overline{\overline{[r]}} = 1$ (r is trivial)

(C_3) $\overline{\overline{[r]}} = 1$ and $\overline{\overline{E}} \geqq n$ and $\overline{\overline{E}} \equiv n \pmod 2$

(C_4) $r \notin R_n(G, E)$

PROPOSITION 3.1. $(i = 0, 1, 2, 3,)$: *If* $\langle G, E, r \rangle$ *satifies* (C_i) *then there is* $\langle G', E'. r' \rangle$ *satisfying* (C_{i+1}) *such that* $\langle G, E, r \rangle \operatorname{ext} \langle G', E', r' \rangle$.

PROOF OF 3.0. Given $\langle G, E, r \rangle$. Let $[r] = \{F_1, \cdots, F_k\}$. If $k = n$ we set $\langle G', E', r' \rangle = \langle G, E, r \rangle$. If $k < n$, we pick one element $c_i \in F_i$ for each $i \leqq k$. Let A be a complete $(n-k)$-graph disjoint to G. Let G' be the sum $G + A$ together with the edges $\{c_i, a\}$, $i \leqq k$, $a \in |A|$. Let $E' = E \cup |A|$ and $r' = r \cup$ (identity on $|A|$). Then $[r'] = n$.

PROOF OF (a). Let $\sigma \in C_n(G')$ such that $\sigma \parallel E = r$. Then $r' \subseteq \sigma \parallel E'$. To verify the converse inclusion assume $x(\sigma \parallel E')y$ and consider the three cases $x, y \in E$, $x, y \in |A|$, $x \in E$ and $y \in |A|$. The last case is impossible since $x r c_i$, some i, and the edge $\{c_i, y\}$ belongs to G'. The first two cases clearly imply $x r' y$. Thus $r' = \sigma \parallel E'$.

PROOF OF (b). Let $\sigma \in C_n(G)$ and $\sigma \parallel E \neq r$. Since $\overline{\overline{A}} = n - k$, there is $\tau : |G'| \to n$ such that $\sigma \subseteq \tau$ and $\tau \upharpoonright |A|$ is one-one and $\tau a \neq \tau c_i$, all $a \in |A|$ and $i \leqq k$. Then $\tau \in C_n(G')$ and $\tau \parallel E' \neq r'$, since $r' \upharpoonright E = r$.

Proof of 3.1. Given $\langle G,E,r\rangle$ satisfying (C_1). Let $[r] = \{F_1,\cdots,F_n\}$. For each $\langle x_1,x_2,\cdots,x_{n-1}\rangle \in F_1 \times F_2 \times \cdots \times F_{n-1}$ we introduce a new vertex $P(x_1,\cdots,x_{n-1})$ and the edges $\{x_i, P(x_1,\cdots,x_{n-1})\}$, $i = 1,2,\cdots,n-1$. G' is the resulting extension of G. Let $E' = F_n \cup \{P(t)\colon t \in F_1 \times \cdots \times F_{n-1}\}$ and $r' = E' \times E'$. Thus $\overline{\overline{[r']}} = 1$. The proof of (a) is immediate.

Proof of (b). Let $\sigma \in C_n(G)$ and $\sigma \| E \neq r$. Then $\sigma \| E \nsubseteq r$ since $\overline{\overline{[r]}} = n$. Therefore there are i,j, $i < j \leqq n$, and $a \in F_i$, $b \in F_j$ such that $\sigma a = \sigma b$.

Case 1. $j = n$. Since every new vertex is joined to only $n-1$ old vertices, σ can be extended to an n-coloring τ of G'. a is ith component of some $t \in F_1 \times \cdots \times F_{n-1}$. Thus $\tau b = \tau a \neq \tau P(t)$. Hence $\tau \| E' \neq r'$ since $b \in E'$ and $P(t) \in E'$.

Case 2. $j < n$. Let $t \in F_1 \times \cdots \times F_{n-1}$ with ith component a and jth component b. Since $\sigma a = \sigma b$ there are at least two choices for $\tau P(t)$. Therefore τ can be chosen such that $\tau \restriction E'$ is non-constant.

Proof of 3.2. Given $\langle E,G,r\rangle$ satisfying (C_2). Choose k such that $\overline{\overline{E}} + k$ is $\geqq n$ and $\equiv n \pmod 2$. We introduce k new vertices $P_1,\cdots,P_k$ and k disjoint complete $(n-1)$-graphs $A_1,\cdots,A_k$ and pick one element $P_0 \in E$. Each vertex of A_i is joined to the two vertices P_i and P_{i-1}, $i = 1,2,\cdots,k$. G' is the resulting extension of G, $E' = E \cup \{P_1,\cdots,P_k\}$, $r' = E' \times E'$. Clearly $\langle G',E',r'\rangle$ is an "ext"-extension satisfying (C_3).

Proof of 3.3. Given $\langle G,E,r\rangle$ satisfying (C_3). Let A be a graph, $\overline{\overline{A}} = \overline{\overline{E}}$ and $|A| \cap |G| = 0$, satisfying (i) and (ii) of Proposition 2. Let g be a one-to-one function from $|A|$ onto E. Let G' be the sum $G + A$ together with the edges $\{a, ga\}$, $a \in |A|$. Let $E' = E$, $r' = r$. Since A is not $(n-1)$-colorable, we have $r \notin R_n(G',E)$, hence (C_4): $r' \notin R_n(G',E')$. Condition (a) holds for the same reason: If $\sigma \in C_n(G')$ then $\sigma \| E \neq r$.

Proof of (b). Given $\sigma \in C_n(G)$ with $\sigma \| E \neq r$. Let $fa = \sigma(ga)$, $a \in |A|$. Then f is non-constant. By Proposition 2(ii) there is $\sigma_1 \in C_n(A)$ with $\sigma_1 a \neq \sigma(ga)$, all $a \in |A|$. Then $\tau = \sigma \cup \sigma_1$ is the required extension of σ.

Proposition 4. *Coloring Extension Lemma for* $K_r = eq_n(E) - \{r\}$

Proof. Given E, r. Let G_0 be the graph on E without any edges. By Proposition 3 and transitivity of "ext" there is $\langle G',E',r'\rangle$ satisfying (C_4) such that $\langle G_0,E,r\rangle \operatorname{ext} \langle G',E',r'\rangle$. Since $r' \notin R_n(G',E')$, (a) implies $r \notin R_n(G',E)$. If

$s \in eq_n(E) - \{r\}$, let $\sigma \in C_n(G_0)$ with $\sigma \| E = s$. By (b), σ can be extended to an n-coloring of G'. Therefore $s \in R_n(G', E)$. Hence $R_n(G', E) = K_r$.

As noted above, the Coloring Extension Lemma follows from Propositions 1 and 4.

Part 2

In order to avoid the axoim of choice we need a uniform way of forming disjoint unions of arbitrary sets of graphs.

The pair $\langle G, E \rangle$ satisfies condition (*) if G is a graph, $E \subseteq |G|$, and $E \subseteq V \times \{0\}$ (V is the universe, 0 the empty set).

Given $\langle G, E \rangle$, let

$$fx = \begin{cases} \langle x, \langle G, E \rangle \rangle, & \text{if } x \in |G| - E \\ x, & \text{if } x \in E. \end{cases}$$

f is one-to-one on $|G|$ since $\langle x, \langle G, E \rangle \rangle \notin E$.

Let $[G, E]$ denote the f-isomorphic image of G.

PROPOSITION 5.

1) $E \subseteq |[G, E]|$ *and* $R_n([G, E], E) = R_n(G, E)$;

2) *If* $\langle G, E \rangle$, $\langle G', E' \rangle$ *both satisfy* (*) *and* $\langle G, E \rangle \neq \langle G', E' \rangle$, *then* $|[G, E]| \cap |[G', E']| = E \cap E'$.

PROOF OF (2). Let $y \in |[G, E]| \cap |G', E']|$. If $y \notin E$ then $y = \langle x, \langle G, E \rangle \rangle$ and therefore $y \notin E'$ since $\langle G, E \rangle \neq 0$ and $E' \subseteq V \times \{0\}$. Hence $y = \langle x', \langle G', E' \rangle \rangle$ which contradicts $\langle G', E' \rangle \neq \langle G, E \rangle$. Therefore $y \in E \cap E'$. The converse inclusion follows from (1).

Let M be a set of pairs $\langle G, E \rangle$ satisfying (*). We define

$$\hat{\Sigma} M = \Sigma\{[G, E]: \langle G, E \rangle \in M\},$$

$$U_M = \bigcup \{E: \exists G(\langle G, E \rangle \in M)\}.$$

PROPOSITION 6. *For M as above*

1) $R_n(\hat{\Sigma} M, U_M) \subseteq \{r \in eq_n(U_M): \forall \langle G, E \rangle \in M \; (r \upharpoonright E \in R_n(G, E))\}$

2) *If M is finite, then the converse inclusion holds too.*

Note. The condition of finiteness is only required to avoid the axiom of choice.

Proof. (1) is immediate from Proposition 5(1). Assume then M finite and $r \in eq_n(U_M)$ such that $r \upharpoonright E \in R_n(G, E)$, all $\langle G, E \rangle \in M$. By 5(1), $r \upharpoonright E \in R_n([G, E], E)$.

Let $\sigma_0\colon U_M \to n$ such that $\sigma_0 \| U_M = r$. For each $\langle G,E\rangle \in M$ we choose an n-coloring $\sigma(G,E)$ of $[G,E]$ which extends $\sigma_0 \restriction E$. Let $\sigma = \bigcup\{\sigma(G,E); \langle G,E\rangle \in M\}$. Then σ is single-valued (Proposition 5(2)). Therefore $\sigma \in C_n(\hat{\Sigma}M)$ and $\sigma \| U_M = r$. Hence $r \in R_n(\hat{\Sigma}M, U_M)$.

THEOREM (of set theory without AC). *There is a function G which assigns to each Boolean algebra B a graph $G(B)$ such that*

1) *if $G(B)$ is 3-colorable then there is a prime ideal in B,*
2) *every finite subgraph of $G(B)$ is 3-colorable.*

COROLLARY. $P_3 \to I$.

PROOF OF THE THEOREM. It suffices to define $G(B)$ for the case $|B| \subseteq V \times \{0\}$. Let $\mathrm{fin}(B)$ denote the set of finite subalgebras of B. If $I \subseteq |B^*| \subseteq |B|$, let $r(B^*,I)$ denote the equivalence relation on $|B^*|$ corresponding to the partition $\{I, |B^*| - I\}$. Let $K(B^*) = \{r(B^*,I)\colon I$ prime ideal in $B^*\}$. Then $K(B^*) \subseteq eq_2(|B^*|)$. Let $M = \{\langle G, |B^*|\rangle\colon B^* \in \mathrm{fin}(B)$ and $|G| \subseteq |B^*| \cup \omega$ and $R_3(G,|B^*|) = K(B^*)\}$.

The graph $G(B) = \hat{\Sigma}M$ then satisfies (1) and (2).

For the proof we recall the following:

a) Every finitely generated Boolean algebra is finite.

b) Every finite Boolean algebra has prime ideals.

c) The restriction of a prime ideal to a subalgebra is a prime ideal of the subalgebra.

d) If $I \subseteq |B|$ and $I \cap |B^*|$ is a prime ideal of B^* for all $B^* \in \mathrm{fin}(B)$, then I is a prime ideal of B.

PROOF OF (1). From the (Corollary to the) Coloring Extension Lemma we get:

(+) For each $B^* \in \mathrm{fin}(B)$ there is G such that $\langle G, |B^*|\rangle \in M$. In particular, $|B| = U_M \subseteq |G(B)|$.

Given a 3-coloring σ of $G(B)$, let $I = \{x \in |B|\colon \sigma x = \sigma 0\}$, where 0 denotes the zero-element of B. We show that I is a prime ideal of B. Let $r = \sigma \| U_M$. Then $r \restriction |B^*| \in K(B^*)$, all $B^* \in \mathrm{fin}(B)$ (Proposition 6(1) and (+)). Therefore, since $I \cap |B^*|$ is the equivalence class of $r \restriction |B^*|$ containing 0, $I \cap |B^*|$ is a prime ideal of B^* for all $B^* \in \mathrm{fin}(B)$. By (d), I is a prime ideal of B.

PROOF OF (2). Let G^* be a finite subgraph of $G(B)$. Let N be a finite subset of M with $G^* \leq \hat{\Sigma}N$. By (a) there is $B_0 \in \mathrm{fin}(B)$ such that $|B^*| \subseteq |B_0|$ for all

B^* occurring in N. Let I_0 be a prime ideal of B_0 ((b)). Let $r_0 = r(B_0, I_0) \restriction U_N$. Then $r_0 \in eq_3(U_N)$, and for all $\langle G, |B^*| \rangle \in N$ we have $r_0 \restriction |B^*| = r(B^*, I_0 \cap |B^*|) \in K(B^*) = R_3(G, |B^*|)$ by (c). Proposition 6(2) yields $r_0 \in R_3(\hat{\Sigma} N, U_N)$. In particular, $\hat{\Sigma} N$ is 3-colorable. Thus G^* is 3-colorable.

PROBLEM. Give a "direct" proof of $P_3 \to P_4$.

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E.T.H. ZÜRICH